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Direct mode summation for the Casimir energy of a solid ball

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Abstract. The Casimir energy of a solid ball placed in an infinite medium is calculated by a direct frequency summation using the contour integration. It is assumed that the permittivity and permeability of the ball and medium satisfy the condition $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$. After deriving the general expression for the Casimir energy, a compact ball is considered when $\xi^2 = (\varepsilon_1 - \varepsilon_2)^2/(\varepsilon_1 + \varepsilon_2)^2 \ll 1$. Calculations are carried out which are of the first order in ξ^2 and take account of the fifth-order terms in the uniform asymptotic expansion of the Bessel functions involved. The implication of the results to attempt to explain sonoluminescence via the Casimir effect is briefly discussed.

1. Introduction

Casimir energy, determined by the first quantum correction to the ground state of a quantum field system with allowance for nontrivial boundary conditions, proves to be essential in many problems of elementary particle theory, in quantum cosmology, and in the physics of condensed matter. However, up to now there has been no universal method for calculating the Casimir effect for arbitrary boundary conditions. It has been done only for simple field configurations of high symmetry: the gap between two plates, sphere, cylinder, wedge and so on. The curvature of the boundary and accounting for the dielectric and magnetic properties of the medium lead to considerable complications. While the attractive force between two uncharged metal plates was calculated by Casimir as far back as 1948 [1], the same effect for a perfectly conducting spherical shell in a vacuum was computed by Boyer only in 1968 [2] (see also the later calculations [3–6]). For an infinitely thin spherical shell separating media with arbitrary dielectric $(\varepsilon_1, \varepsilon_2)$ and magnetic (μ_1, μ_2) characteristics, the problem is as yet unsolved [7–10]. The main difficulty is the lack of a consistent method for removing the divergences. Besides an attempt to revive the quasiclassical model of an extended electron proposed by Casimir [11], interest in this problem has also been raised by investigations of bag models in hadron physics [12–14] and the recent search for a mechanism of sonoluminescence [15].

In this paper we calculate the Casimir energy of a solid ball by making use of the direct summation of the eigenfrequencies of a vacuum electromagnetic field by contour

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integration [16, 17]. A definite advantage of this method, compared with the Green's function technique employed in [7–9, 18], is its simplicity and visualization. We consider a compact ball placed in an infinite medium when $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$. This condition enables one to treat the divergencies analogously to the case of a perfectly conducting spherical shell [6]. Upon deriving the general expression for Casimir energy, we address ourselves to the case of a compact ball with $\xi^2 \ll 1$, $\xi = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$. The calculations here are of the first order in ξ^2 and take account of the fifth-order terms in the uniform asymptotic expansion of the Bessel functions involved. In this way we attain some generalization and refinement of the results obtained for this problem previously [18].

The layout of this paper is as follows. In section 2 we derive a general expression for the Casimir energy of a solid ball in an infinite surrounding under the condition $\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = c^{-2}$, where c is an arbitrary constant not necessarily equal to one (it is the light velocity in the medium), the mode-by-mode summation of eigenfrequencies being used. In section 3 the Casimir energy of a compact ball is calculated when $\xi^2 \ll 1$. The implication of the obtained result to the Schwinger attempt to explain the sonoluminescence via the Casimir effect is also considered. In section 4 the results of the paper are briefly discussed. Dispersive effects are ignored in our paper.

2. Casimir energy of a solid ball under the condition $\varepsilon_1\mu_1 = \varepsilon_2\mu_2$

Let us consider the Casimir theory of a solid ball of radius a, consisting of a material which is characterized by permittivity ε_1 and permeability μ_1. We assume that the ball is placed in an infinite medium with permittivity ε_2 and permeability μ_2. We also suppose that the conductivity of the ball material and its surroundings is equal to zero.

In our consideration the main part will be played by equations determining the eigenfrequencies ω of the electromagnetic oscillations for this configuration [19]. It is convenient to rewrite these equations in terms of the Riccati–Bessel functions

$$\tilde{s}_l(x) = x j_l(x) \qquad \tilde{e}_l(x) = x h_l^{(1)}(x) \tag{2.1}$$

where $j_l(x) = \sqrt{\pi/2x}\, J_{l+1/2}(x)$ is the spherical Bessel function and $h_l^{(1)}(x) = \sqrt{\pi/2x}\, H_{l+1/2}^{(1)}(x)$ is the spherical Hankel function of the first kind. For the TE-modes the frequency equation reads

$$\Delta_l^{\mathrm{TE}}(a\omega) \equiv \sqrt{\varepsilon_1\mu_2}\,\tilde{s}_l'(k_1 a)\tilde{e}_l(k_2 a) - \sqrt{\varepsilon_2\mu_1}\,\tilde{s}_l(k_1 a)\tilde{e}_l'(k_2 a) = 0 \tag{2.2}$$

where $k_i = \sqrt{\varepsilon_i\mu_i}\,\omega$, $i = 1, 2$, are the wavenumbers inside and outside the ball, respectively; the primes represent the differentiation with respect to the argument ($k_1 a$ or $k_2 a$) of the corresponding Riccati–Bessel function. The frequencies of the TM-modes are determined by

$$\Delta_l^{\mathrm{TM}}(a\omega) \equiv \sqrt{\varepsilon_2\mu_1}\,\tilde{s}_l'(k_1 a)\tilde{e}_l(k_2 a) - \sqrt{\varepsilon_1\mu_2}\,\tilde{s}_l(k_1 a)\tilde{e}_l'(k_2 a) = 0. \tag{2.3}$$

The orbital quantum number l in (2.2) and (2.3) assumes the values $1, 2, \ldots$. Under mutual change $\varepsilon_i \leftrightarrow \mu_i$, $i = 1, 2$, frequency equations (2.2) and (2.3) transform into each other.

It is worth noting that the frequencies of the electromagnetic oscillations determined by equations (2.2) and (2.3) are the same inside and outside the ball. This is in contrast to the case of a perfectly conducting spherical shell in a vacuum [6], where eigenfrequencies inside and outside the shell are determined by different equations [19]. The physical reason for this is that photons do not perform work when passing through the boundary at $r = a$. Here we disregard the variation of the velocity of light due to the higher radiative corrections to the vacuum energy (the Scharnhorst effect [20]) because they are vanishingly small.

As usual we define the Casimir energy by the formula

$$E = \frac{1}{2}\sum_{p}(\omega_p - \bar{\omega}_p) \tag{2.4}$$

where ω_p are the roots of equations (2.2) and (2.3) and $\bar{\omega}_p$ are the same roots under condition $a \to \infty$. Here p is a collective index that stands for a complete set of indices for the roots of equations (2.2) and (2.3). The sum in (2.4) obviously diverges, and it will require appropriate regularizations (see below).

Denoting the roots of equations (2.2) and (2.3) by $\omega_{nl}^{(1)}$ and $\omega_{nl}^{(2)}$ respectively, we can cast equation (2.4) in the explicit form

$$E = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{n=1}^{\infty}(\omega_{nl}^{(\alpha)} - \bar{\omega}_{nl}^{(\alpha)}) = \sum_{l=1}^{\infty} E_l \tag{2.5}$$

where the notation

$$E_l = (l + \tfrac{1}{2})^{1-s}\sum_{\alpha=1}^{2}\sum_{n=1}^{\infty}(\omega_{nl}^{(\alpha)} - \bar{\omega}_{nl}^{(\alpha)}) \tag{2.6}$$

is introduced. Here we have taken into account that the eigenfrequencies $\omega_{nl}^{(\alpha)}$ do not depend on the azimuthal quantum number m. The parameter s is introduced to regularize the sum over l. In the intermediate calculations we treat this parameter as large enough to make the sum over l convergent. The regularization will be removed at the end of our consideration by putting $s = 0$. For partial energies E_l we use representation in terms of the contour integral provided by the Cauchy theorem [21]

$$E_l = \frac{(l+1/2)^{1-s}}{2\pi \mathrm{i}}\oint_C \mathrm{d}z\, z\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\Delta_l^{\mathrm{TE}}(az)\Delta_l^{\mathrm{TM}}(az)}{\Delta_l^{\mathrm{TE}}(\infty)\Delta_l^{\mathrm{TM}}(\infty)} \tag{2.7}$$

where the contour C surrounds, counterclockwise, the roots of the frequency equations in the right half-plane. Location of the roots of equations (2.2) and (2.3) enables one to deform the contour C into a segment of the imaginary axis $(-\mathrm{i}\Lambda, \mathrm{i}\Lambda)$ and a semicircle of radius Λ in the right half-plane. At a given value of Λ a finite number of the roots of the frequency equations is taken into account. Thus Λ plays the role of a regularization parameter for the initial sum over n in equation (2.6) which should be subsequently taken to infinity. In this limit the contribution of the semicircle of radius Λ into integral (2.7) vanishes. From physical considerations it is clear that multiplier z in (2.7) is understood to be the $\lim_{\mu\to 0}\sqrt{z^2+\mu^2}$, where μ is the photon mass. Therefore in the integral along the segment $(-\mathrm{i}\Lambda, \mathrm{i}\Lambda)$ we can integrate once by parts, the nonintegral terms being cancelled. As a result, after the subtraction according to (2.4) and removal of the regularization $(\Lambda \to \infty)$ one obtains

$$E_l = \frac{(l+\frac{1}{2})^{1-s}}{\pi a}\int_0^{\infty}\mathrm{d}y\,\ln\frac{\Delta_l^{\mathrm{TE}}(\mathrm{i}y)\Delta_l^{\mathrm{TM}}(\mathrm{i}y)}{\Delta_l^{\mathrm{TE}}(\mathrm{i}\infty)\Delta_l^{\mathrm{TM}}(\mathrm{i}\infty)}. \tag{2.8}$$

Now we need the modified Riccati–Bessel functions

$$s_l(x) = \sqrt{\frac{\pi x}{2}}I_\nu(x) \qquad e_l(x) = \sqrt{\frac{2x}{\pi}}K_\nu(x) \qquad \nu = l + \tfrac{1}{2} \tag{2.9}$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions [22]. With allowance for the asymptotics of $s_l(x)$ and $e_l(x)$ at $x \to \infty$

$$s_l(x) \simeq \tfrac{1}{2}\mathrm{e}^{x} \tag{2.10}$$

$$e_l(x) \simeq \mathrm{e}^{-x} \tag{2.11}$$

equation (2.8) can be rewritten as

$$E_l = \frac{(l +' \frac{1}{2})^{1-s}}{\pi a} \int_0^\infty \mathrm{d}y \ln \left\{ \frac{4\mathrm{e}^{-2(q_1-q_2)}}{(\sqrt{\varepsilon_1\mu_2} + \sqrt{\varepsilon_2\mu_1})^2} \left[\sqrt{\varepsilon_1\varepsilon_2\mu_1\mu_2}((s_l'(q_1)e_l(q_2))^2 \right.\right.$$
$$\left.\left. + (s_l(q_1)e_l'(q_2))^2 - (\varepsilon_1\mu_2 + \varepsilon_2\mu_1)s_l(q_1)s_l'(q_1)e_l(q_2)e_l'(q_2) \right] \right\} \tag{2.12}$$

where $q_i = \sqrt{\varepsilon_i\mu_i}\, y$, $i = 1, 2$. We shall use this general equation in the next section but here we address the special case when the condition

$$\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = c^{-2} \tag{2.13}$$

is fulfilled. Here c is an arbitrary positive constant (the light velocity in the medium). Physical implications of this condition at $c = 1$ can be found in [23]. Now equation (2.12) assumes the form

$$E_l = \frac{c(l + \frac{1}{2})^{1-s}}{\pi a} \int_0^\infty \mathrm{d}y \ln \left\{ \frac{4}{\varepsilon + \varepsilon^{-1} + 2} [(s_l'(y)e_l(y))^2 + (s_l(y)e_l'(y))^2 \right.$$
$$\left. -(\varepsilon + \varepsilon^{-1})s_l(y)s_l'(y)e_l(y)e_l'(y)] \right\} \tag{2.14}$$

where $\varepsilon = \varepsilon_1/\varepsilon_2$. The argument of the logarithm in (2.14) can be transformed, if the following two equalities for the functions $s_l(y)$ and $e_l(y)$

$$s_l'(y)e_l(y) - s_l(y)e_l'(y) = 1 \tag{2.15}$$
$$s_l'(y)e_l(y) + s_l(y)e_l'(y) = (s_l(y)e_l(y))' \tag{2.16}$$

are taken into account. It gives

$$E_l = \frac{c(l + \frac{1}{2})^{1-s}}{\pi a} \int_0^\infty \mathrm{d}y \ln\{1 - \xi^2[(s_l(y)e_l(y))']^2\} \tag{2.17}$$

where

$$\xi = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \qquad \varepsilon_i\mu_i = c^{-2} \qquad i = 1, 2. \tag{2.18}$$

Thus, for a ball with a vacuum on the outside, $\xi = (1 - \mu)/(1 + \mu) = (\varepsilon - 1)/(\varepsilon + 1)$ and $c = 1$. It is important to note that under the condition (2.13) the general structure of the divergences in the considered problem proves to be the same as in the case of a perfectly conducting spherical shell [6]. The expression (2.17) agrees with the results obtained in [8, 24], if one performs a partial integration of the expression for E given in these references and puts the cut-off parameter δ equal to zero. If $\xi^2 = 1$ then equation (2.17) turns into the analogous expression for the perfectly conducting spherical shell in a vacuum [3, 6].

We remark that in a previous paper [18] an expression for the Casimir energy was calculated that is seemingly in conflict with equation (2.17). Namely, equation (2.42) in that paper corresponds to the following expression for E_l, assuming $\mu_1 = \mu, \mu_2 = 1$ as above, and putting the cut-off parameter equal to zero:

$$E_l = -\frac{(\mu - 1)^2}{\pi a}\nu \int_0^\infty \mathrm{d}x \frac{s_l s_l' e_l e_l'}{D_l \tilde{D}_l} x \frac{\mathrm{d}}{\mathrm{d}x} \ln(1 - \lambda_l^2). \tag{2.19}$$

Here $\lambda_l = (s_l e_l)'$, and D_l, $\tilde{D}_l$ are defined by

$$D_l(x) = \mu s_l(x)e_l'(x) - s_l'(x)e_l(x) \tag{2.20}$$
$$\tilde{D}_l(x) = \mu s_l'(x)e_l(x) - s_l(x)e_l'(x). \tag{2.21}$$

It turns out, however, that these two equations (2.17) and (2.19) are in agreement. To show the equivalence is not quite trivial, but follows after some algebra taking into account the derivatives of the logarithms and the Wronskian (2.15). We omit the details here. The expression (2.17) is the most convenient form to work with, and we use it in the following where we turn to the special case of a dilute medium satisfying the condition $\varepsilon\mu = 1$. We emphasize that the expression (2.17) is general, making no restriction at all on the magnitude of μ_1/μ_2.

3. Casimir energy of a compact ball for $|\xi| \ll 1$

Now we address ourselves to consideration of the Casimir energy of a compact ball when

$$|\xi| \ll 1. \tag{3.1}$$

That means, to the lowest order in ξ,

$$\ln(1-\xi^2\lambda_l^2) \simeq -\xi^2\lambda_l^2 \tag{3.2}$$

which reflects a general property of all Casimir calculations in dilute media: the lowest order correction for all physical quantities is proportional to the *square* of the susceptibility (electric or magnetic). We shall henceforth work only to the second order in ξ. From (2.17) and (3.2) we then get for the Casimir energy

$$E = -\frac{c\xi^2}{\pi a}\sum_{l=1}^{\infty}\nu^{1-s}\int_0^{\infty}\mathrm{d}x\,\lambda_l^2(x). \tag{3.3}$$

To perform the summation with respect to l one should know the behaviour of (3.3) when $l \to \infty$. We now invoke the following useful expansion at $\nu \to \infty$, which was worked out by one of us some time ago [25]

$$\begin{aligned}\lambda_l(x) = (s_l(x)e_l(x))' = \frac{t^3}{2\nu}\bigg[1 &- \frac{1}{8\nu^2}(2-27t^2+60t^4-35t^6)\\ &-\frac{1}{128\nu^4}(108t^2-3615t^4+21\,420t^6-47\,250t^8\\ &+44\,352t^{10}-15\,015t^{12})+\mathcal{O}(1/\nu^6)\bigg].\end{aligned} \tag{3.4}$$

Here, $t(z) = (1+z^2)^{-1/2}$, $z = x/\nu$. This expression is based upon the uniform with respect to z asymptotic expansions (UAE) for the modified Bessel functions [22] at $l \to \infty$. From (3.4) we calculate

$$\begin{aligned}\lambda_l^2(x) = \frac{t^6}{4\nu^2}\bigg[1 &- \frac{1}{4\nu^2}(2-27t^2+60t^4-35t^6)+\frac{1}{16\nu^4}(1-54t^2+1146t^4-6200t^6\\ &+13185t^8-12\,138t^{10}+4060t^{12})+\mathcal{O}(1/\nu^6)\bigg]\end{aligned} \tag{3.5}$$

which can now be inserted into equation (3.3). From the integral representation of the beta function [22] $B(q,p) = \Gamma(q)\Gamma(p)/\Gamma(q+p)$ we derive the formula

$$\int_0^{\infty} t^p(z)\,\mathrm{d}z = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \tag{3.6}$$

which is useful in the present context. After some calculation we obtain

$$E = -\frac{3\xi^2 c}{64a}\left[\sum_{l=1}^{\infty}\nu^{-s} - \frac{9}{128}\sum_{l=1}^{\infty}\nu^{-2-s} + \frac{423}{16\,384}\sum_{l=1}^{\infty}\nu^{-4-s} + \mathcal{O}\left(\frac{1}{\nu^{6+s}}\right)\right]. \tag{3.7}$$

For $s > 1$ the sums in (3.7) are expressed in terms of the Riemann zeta function. This technique [26] turns out to be most useful in all Casimir problems involving nondispersive media. In practical calculations the only formula needed is

$$\sum_{l=0}^{\infty}\nu^{-s} = (2^s - 1)\zeta(s) \tag{3.8}$$

from which it follows that

$$E = -\frac{3\xi^2 c}{64a}\left\{[(2^s-1)\zeta(s) - 2^s] - \frac{9}{128}[(2^{s+2}-1)\zeta(s+2) - 2^{s+2}] \right.$$
$$\left. + \frac{423}{16\,384}[(2^{s+4}-1)\zeta(s+4) - 2^{s+4}] + \mathcal{O}\left(\frac{1}{\nu^{s+6}}\right)\right\}. \tag{3.9}$$

To remove the regularization one should put in this expression $s = 0$. Finally for the Casimir energy of a ball we get, omitting the remainder in (3.9),

$$E = \frac{3\xi^2}{64a}\left[1 + \frac{9}{128}\left(\frac{1}{2}\pi^2 - 4\right) - \frac{423}{16\,384}\left(\frac{1}{6}\pi^4 - 16\right)\right]. \tag{3.10}$$

The energy is positive, corresponding to a *repulsive* surface force. Remember, though, that we are working here with the *nondispersive* theory only.

The structure of the three different terms in (3.10) is the following. The first term stems from the order $1/\nu$ in the uniform asymptotic expansion for the Bessel functions. Numerically, the three terms between square brackets in (3.10) are $[1+0.065\,73-0.006\,10]$. Thus the second term, stemming from the order $1/\nu^3$ in the UAE, describes a repulsive correction of about 6.6%. Finally the third term, stemming from the order $1/\nu^5$ in the UAE, describes a 0.6% attractive correction. We have thus improved the calculations in [18, 25] by four orders in magnitude. The next correction, not included here, is of order $1/\nu^7$ in the UAE.

In conclusion we address the consideration of a dielectric ball, when $\mu_1 = \mu_2 = 1$ and its permittivity and that of the surrounding differ slightly ($\varepsilon_1 + \varepsilon_2 = 2\varepsilon, \varepsilon_1 - \varepsilon_2 = 2\triangle\varepsilon, |\triangle\varepsilon|/\varepsilon \ll 1$). Under these conditions one can simplify the general formula (2.12) putting there $q_1 = q_2$ in the arguments of the Bessel functions and of the exponential. After making use of equations (2.15) and (2.16) we again arrive at (2.17), where now

$$\xi^2 = \left(\frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}\right)^2 \simeq \frac{1}{4}\left(\frac{\triangle\varepsilon}{\varepsilon}\right)^2 \qquad \text{and} \qquad c = \frac{1}{\sqrt{\varepsilon}}.$$

After the summation over l using the zeta function technique we arrive at

$$E \simeq 3\xi^2/(64a). \tag{3.11}$$

This formula gives the Casimir energy of a nonmagnetic dilute dielectric ball or a spherical cavity in an infinite surrounding. Take, as an example, $|\xi| = 0.1$, $a = 4 \times 10^{-4}$ cm. Then $E \simeq 2\times10^{-5}$ eV. This is markedly smaller than the amount of energy ($\sim$10 MeV) emitted in a sonoluminescent flash. Furthermore, the Casimir energy (3.11), being positive, increases when the radius of the ball decreases. The latter eliminates completely the possibility of explaining, via the Casimir effect, sonoluminescence for bubbles in a liquid. As known [27], emission of light takes place at the end of the bubble collapse. Recent important experimental studies have measured the duration of the sonoluminescence flash [28]. In view of all this it is difficult to imagine that the Casimir effect, at least in its nondispersive version, should be important for the sonoluminescence phenomenon. Comparing our result for the Casimir energy of a dilute dielectric ball (3.11) with other calculations of this energy we see that it is close to equations (3.17) and (3.26) in [8] differing only by the factor $9\pi/46 \simeq 0.6$. This is important for justification of our consideration because equations (3.17) and (3.26) in [9] have been derived in the framework of an absolutely different but physically clear approach—by a direct summation of the van der Waals forces. Our result (3.11) differs by the factor $-\frac{3}{4}$ from equation (7.5) in [8] and by the dependence on $\Delta\varepsilon$ from the calculation in [32].

4. Conclusion

The method used in this paper for calculating the Casimir energy E by means of the contour integral (2.7) has a rather long history (beginning essentially with Boyer [2]) and proves to be very convenient and effective. As known, there are in principle at least two different methods for calculating E: one can follow a local approach, implying use of the Green's function to find the energy density (or the surface force density), or one can sum the eigenfrequencies directly. Equation (2.7) means that we have adopted the latter method here. The Cauchy integral formula turns out to be the most useful in other contexts also, such as in the calculation of the Casimir energy for a piece-wise uniform relativistic string [29]. A survey of this subject can be found in [30]. The great advantage of the method is that the multiplicity of zeros in the dispersion function is automatically taken care of, i.e. one does not have to plug in the degeneracy in the formalism by hand.

A remarkable feature of the approach in hand is that the ultimate formula for the Casimir energy has the form of the spectral representation, i.e. of an integral with respect to frequency between the limits $(0, \infty)$ of a smooth function, spectral density. Evidently, for physical applications one needs to know the frequency range which gives the main contribution into the spectral density. An example of this representation for the partial energies E_l is equation (2.17), where the substitution $y = \omega a$ should be made. As shown above, the partial energies E_l decrease rapidly as l increases. Therefore the most interesting are the first few values of l. As one might expect, the spectral density is different from zero when $\omega a \simeq 1$. Keeping in mind the search for the origin of the sonoluminescence we put [8, 27] $a = 4 \times 10^{-4}$ cm. Then the wavelength of the photon in question turns out to be 25.0×10^{-4} cm, i.e. this radiation belongs in the infrared region, while in experiments on sonoluminescence blue light is observed [27]. This fact also argues against the possibility of explaining the sonoluminescence by the Casimir effect.

It is worth noting that the spectral distribution of the Casimir energy is hardly discussed in the literature whereas the space density of this energy has been investigated in detail (see, for example [18]). From the physical point of view the space density and spectral density of energy in this problem should be treated on the same footing. One should remember here the treatment of the Casimir effect as a manifestation of the fluctuations of the vacuum fields [31], these fluctuations occurring in space and time simultaneously.

It should be emphasized that in this paper we have neglected the dispersion effects when calculating the Casimir energy. The importance of this point has been demonstrated in [32]. As for the elucidation of the sonoluminescence origin, we have to stress once more that in our consideration we have contented ourselves with the static Casimir effect only.

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